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Pseudo-Effect Algebras as Total Algebras

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Abstract It is shown that every pseudo-effect algebra can be organized as a total algebra. In general, this total algebra is not unique, but it always determines the original partial addition. Special attention is paid to lattice-ordered pseudo-effect algebras for which the total operations can be defined in a unique natural way and this allows one to characterize pairs of compatible elements.

Keywords Pseudo-effect algebra · Commutative directoid · Antiautomorphism · Compatible elements

Introduction

Effect algebras introduced by Foulis and Bennett in 1994, as well as their equivalent counterpart D-posets independently introduced by Kôpka and Chovanec also in 1994, are commonly known quantum structures. In 2001, Dvurečenskij and Vetterlein [4, 5] established the so-called *pseudo-effect algebras* as a non-commutative generalization of effect algebras. Roughly speaking, a pseudo-effect algebra is a partial algebra $(E, +, 0, 1)$ where $+$ is a partial binary operation on E satisfying certain axioms which are similar to those of effect algebras and guarantee that $(E, +, 0, 1)$ is an effect algebra provided that $+$ is commutative. (For the exact definition see Sect. 2.)

The present paper is a continuation of [2] where we have proved that for every effect algebra $(E, +, 0, 1)$ there exists a total algebra $(E, \oplus, \neg, 0, 1)$ of type $(2, 1, 0, 0)$ such that the restriction of $\oplus$ to the pairs (x, y) with $x \leq \neg y$ is just $+$. Thus given a pseudo-effect algebra $(E, +, 0, 1)$ we construct a total algebra from which the original partial operation $+$ can be

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recovered. We start with structures that we call posets with sectional antiautomorphisms. It is shown in Sect. 1 that it is always possible to define total operations that determine the underlying order as well as the sectional antiautomorphisms. In Sect. 2 we first characterize pseudo-effect algebras within posets with sectional antiautomorphisms and then apply the results of Sect. 1 in order to make an arbitrary pseudo-effect algebra into a total algebra. Finally, in Sect. 3 we focus on compatibility in lattice-ordered pseudo-effect algebras.

1 Posets with Sectional Antiautomorphisms

Throughout, by an *antiautomorphism* on a poset we mean a bijection β with the property that $x \leq y$ iff $\beta(x) \geq \beta(y)$ for all x, y.

Definition 1 A *poset with sectional antiautomorphisms* is a structure $(P, \leq, (\gamma_p, \delta_p)_{p\in P}, 0, 1)$ where $(P, \leq)$ is a poset with least element 0 and greatest element 1 and, for each $p \in P$, γ_p and δ_p are antiautomorphisms on the section $[p, 1] = \{x \in P\colon\ p \leq x\}$ such that the inverse of γ_p is δ_p. If $(P, \leq)$ is a lattice, then we call $(P, \leq, (\gamma_p, \delta_p)_{p\in P}, 0, 1)$ a *lattice with sectional antiautomorphisms*.

Let $(P, \leq, (\gamma_p, \delta_p)_{p\in P}, 0, 1)$ be a poset with sectional antiautomorphisms. Assuming for the moment that $(P, \leq)$ is a lattice with the associated join operation $\vee$, we may define two total binary operations, which we denote by $\to$ and $\rightsquigarrow$, as follows:

$$x \to y := \gamma_y(x \vee y) \quad \text{and} \quad x \rightsquigarrow y := \delta_y(x \vee y). \tag{1}$$

It can easily be seen that these 'arrows' capture all we need to know about the initial lattice with sectional antiautomorphisms:

(a) $x \vee y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y$,
(b) $x \leq y$ iff $x \to y = 1$ iff $x \rightsquigarrow y = 1$,
(c) for every a and $x \in [a, 1]$, $\gamma_a(x) = x \to a$ and $\delta_a(x) = x \rightsquigarrow a$.

We have defined $x \to y$ and $x \rightsquigarrow y$ respectively as $\gamma_y(u)$ and $\delta_y(u)$ where $u = x \vee y$, but in order for (1) to be correct it is not necessary that u is the supremum of $\{x, y\}$; it suffices to require that u is *some* upper bound of $\{x, y\}$. Thus our idea is to equip P with a 'join-like' operation $\sqcup$ so that we could introduce the 'arrows' by setting

$$x \to y := \gamma_y(x \sqcup y) \quad \text{and} \quad x \rightsquigarrow y := \delta_y(x \sqcup y). \tag{2}$$

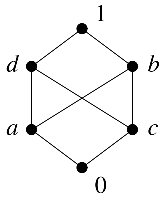
Since $(P, \leq)$ is a bounded poset, every two-element subset of P has an upper bound and we are allowed to define $\sqcup$ as follows:

$$x \sqcup y = y \sqcup x := \begin{cases} x \vee y & \text{if } x \vee y = \sup\{x, y\} \text{ exists,} \\ \text{some upper bound of } \{x, y\} & \text{otherwise.} \end{cases} \tag{3}$$

The groupoid $(P, \sqcup)$ thus obtained is a commutative directoid in the sense of Ježek and Quackenbush [7]. Let us recall that a *commutative directoid* is a commutative, idempotent groupoid $(D, \cdot)$ satisfying the identity $x \cdot ((x \cdot y) \cdot z) = (x \cdot y) \cdot z$. As in semilattices the stipulation $x \leq y$ iff $x \cdot y = y$ [resp. $x \leq y$ iff $x \cdot y = x$] defines a partial order on D such that for every $x, y \in D$, $x \cdot y$ is an upper [resp. lower] bound of $\{x, y\}$. Conversely, every upwards [resp. downwards] directed poset could be made into a commutative directoid by letting $x \cdot y = y \cdot x$ be the greater [resp. smaller] of x, y provided that x, y are comparable, and some common upper [resp. lower] bound otherwise.

Observe that in general $x \cdot y$ need not be the supremum [resp. infimum] of $\{x, y\}$ even though it exists, but in (3) we insist on $x \sqcup y$ being $\sup\{x, y\}$ whenever it exists.

Example 1 Let $(P, \leq, (\gamma_p, \delta_p)_{p\in P}, 0, 1)$ be the poset



with the following sectional antiautomorphisms:

$$\begin{array}{llll} \gamma_0\colon a \mapsto b \mapsto c \mapsto d \mapsto a & \text{and} & \delta_0\colon a \mapsto d \mapsto c \mapsto b \mapsto a & \text{in } [0,1], \\ \gamma_a = \delta_a\colon b \mapsto b, d \mapsto d & & & \text{in } [a,1], \\ \gamma_c = \delta_c\colon b \mapsto d \mapsto b & & & \text{in } [c,1]. \end{array}$$

Of course, in each section $[p, 1]$ these maps switch p and 1, and the sections $[b, 1]$ and $[d, 1]$ admit trivial antiautomorphisms only. As $\sup\{a, c\}$ does not exist, $a \sqcup c$ may be equal to b, d or 1. We then have:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	1	x	1	1
b	c	b	1	d	d	1
c	d	y	1	1	1	1
d	a	d	b	b	1	1
1	0	a	b	c	d	1

$\rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	x	1	1
b	a	b	1	d	d	1
c	b	y	1	1	1	1
d	c	d	b	b	1	1
1	0	a	b	c	d	1

where

$$x = \begin{cases} d & \text{if } a \sqcup c = b, \\ b & \text{if } a \sqcup c = d, \\ c & \text{if } a \sqcup c = 1, \end{cases} \qquad y = \begin{cases} b & \text{if } a \sqcup c = b, \\ d & \text{if } a \sqcup c = d, \\ a & \text{if } a \sqcup c = 1. \end{cases}$$

This example shows that there are possibly many different ways of defining the operations $\to, \rightsquigarrow$, depending on $\sqcup$. However, it is worth observing that $\to, \rightsquigarrow$ give essentially the same information about the original structure as the arrows in case of lattices:

(a) $x \sqcup y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y$,
(b) $x \leq y$ (i.e., $x \sqcup y = y$) iff $x \to y = 1$ iff $x \rightsquigarrow y = 1$,
(c) $\gamma_a(x) = x \to a$ and $\delta_a(x) = x \rightsquigarrow a$ for $x \in [a, 1]$.

Proposition 1 *Let $\mathcal{P} = (P, \leq, (\gamma_p, \delta_p)_{p\in P}, 0, 1)$ be a poset with sectional antiautomorphisms equipped with the operation $\sqcup$ defined according to* (3). *Then the algebra $\mathcal{P}^A =$*

$(P, \to, \rightsquigarrow, 0, 1)$ *with* $\to, \rightsquigarrow$ *given by* (2) *satisfies the identities*

$$1 \to x = x = 1 \rightsquigarrow x, \tag{4}$$

$$x \Cup y = y \Cup x = (y \rightsquigarrow x) \to x = (x \rightsquigarrow y) \to y, \tag{5}$$

$$x \to ((x \Cup y) \Cup z) = 1, \tag{6}$$

$$(((x \Cup y) \Cup z) \to x) \to (y \to x) = 1, \tag{7}$$

$$(((x \Cup y) \Cup z) \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow x) = 1, \tag{8}$$

$$0 \to x = 1, \tag{9}$$

where the supplementary term operation $\Cup$ *is defined by*

$$x \Cup y := (x \to y) \rightsquigarrow y. \tag{10}$$

Proof Straightforward; it suffices to note that $x \Cup y = x \sqcup y$ and apply the above observations (a), (b) and (c). □

Remark 1 Let $\mathcal{P}$ and $\mathcal{P}^A$ be as in Proposition 1. If $(P, \leq)$ is a lattice, then $x \Cup y = x \sqcup y = \sup\{x, y\}$ for all $x, y \in P$, and $\mathcal{P}^A$ also satisfies the identities

$$((y \Cup z) \to x) \to (y \to x) = 1, \tag{11}$$

$$((y \Cup z) \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow x) = 1, \tag{12}$$

which in general do not hold true in algebras derived from *posets* with sectional antiautomorphisms. For instance, if we put $a \sqcup c = b$ in Example 1, then $c \Cup d = c \sqcup d = d$ and $((c \Cup d) \to a) \to (c \to a) = d \to b = b \neq 1$.

Proposition 2 *Let* $\mathcal{A} = (A, \to, \rightsquigarrow, 0, 1)$ *be an algebra of type* $(2, 2, 0, 0)$ *satisfying* (4)–(9) *and let* $\Cup$ *be the term operation defined by* (10). *Then* $(A, \Cup)$ *is a commutative directoid whose underlying order is given by* $x \leq y$ *iff* $x \to y = 1$ *iff* $x \rightsquigarrow y = 1$, *and for each* $a \in A$, *the maps* $\gamma_a, \delta_a : [a, 1] \to [a, 1]$ *defined by* $\gamma_a(x) = x \to a$ *and* $\delta_a(x) = x \rightsquigarrow a$ *are mutually inverse antiautomorphisms on the section* $[a, 1]$. *Thus* $\mathcal{A}^P = (A, \leq, (\gamma_a, \delta_a)_{a \in A}, 0, 1)$ *is a poset with sectional antiautomorphisms. Moreover, we have* $\gamma_y(x \Cup y) = x \to y$ *and* $\delta_y(x \Cup y) = x \rightsquigarrow y$ *for all* $x, y \in A$.

Proof We first observe that $(x \Cup 1) \Cup 1 = (1 \Cup x) \Cup 1 = 1 \to ((1 \Cup x) \Cup 1) = 1$ by (5), (4) and (6). Hence, using (7) and (8), we have $1 = (((x \Cup 1) \Cup 1) \to x) \to (1 \to x) = (1 \to x) \to (1 \to x) = x \to x$ and $1 = (((x \Cup 1) \Cup 1) \rightsquigarrow x) \rightsquigarrow (1 \rightsquigarrow x) = (1 \rightsquigarrow x) \rightsquigarrow (1 \rightsquigarrow x) = x \rightsquigarrow x$. It follows that $x \Cup x = (x \to x) \rightsquigarrow x = 1 \rightsquigarrow x = x$. Trivially, $\Cup$ is commutative, and by (6) and (4) we have $x \Cup ((x \Cup y) \Cup z) = (x \to ((x \Cup y) \Cup z)) \rightsquigarrow ((x \Cup y) \Cup z) = 1 \rightsquigarrow ((x \Cup y) \Cup z) = (x \Cup y) \Cup z$. Hence $(A, \Cup)$ is a commutative directoid. Its top element is 1 since $x \Cup 1 = 1 \Cup x = (1 \to x) \rightsquigarrow x = x \rightsquigarrow x = 1$ for all $x \in A$, and the bottom element is 0 since $0 \Cup x = (0 \to x) \rightsquigarrow x = 1 \rightsquigarrow x = x$ for all $x \in A$.

Further, the underlying order $\leq$ of $(A, \Cup)$ is given by

$$x \leq y \quad \Leftrightarrow \quad x \to y = 1 \quad \Leftrightarrow \quad x \rightsquigarrow y = 1.$$

Indeed, if $x \leq y$, i.e., $x \Cup y = y$, then $(y \Cup x) \Cup y = y$ and (7) yields $1 = (((y \Cup x) \Cup y) \to y) \to (x \to y) = (y \to y) \to (x \to y) = 1 \to (x \to y) = x \to y$; and conversely, if $x \to$

$y = 1$, then $x \Cup y = (x \to y) \rightsquigarrow y = 1 \rightsquigarrow y = y$, so $x \leq y$. The equivalence $x \leq y \Leftrightarrow x \rightsquigarrow y = 1$ can be verified in a similar way.

It is now obvious that (7) and (8) can be rewritten in the following form:

$$x \leq y \leq z \quad \Rightarrow \quad z \to x \leq y \to x \quad \text{and} \quad z \rightsquigarrow x \leq y \rightsquigarrow x. \tag{13}$$

Clearly, (13) entails that for every $a \in A$ both $\gamma_a : x \mapsto x \to a$ and $\delta_a : x \mapsto x \rightsquigarrow a$ are well-defined antitone maps from $[a, 1] = \{x \in A : a \leq x\}$ into itself. Moreover, for each $x \in [a, 1]$, $\gamma_a(\delta_a(x)) = (x \rightsquigarrow a) \to a = x \Cup a = x$ and $\delta_a(\gamma_a(x)) = (x \to a) \rightsquigarrow a = x \Cup a = x$. Thus γ_a, δ_a are antiautomorphisms which are inverses of one another.

Finally, $\gamma_y(x \Cup y) = (x \Cup y) \to y = ((x \to y) \rightsquigarrow y) \to y = (x \to y) \Cup y = x \to y$ and likewise $\delta_y(x \Cup y) = (x \Cup y) \rightsquigarrow y = ((x \rightsquigarrow y) \to y) \rightsquigarrow y = (x \rightsquigarrow y) \Cup y = x \rightsquigarrow y$. □

Remark 2 If $\mathcal{A}$ satisfies the identities (11) and (12), then $\mathcal{A}^P$ is a lattice with sectional antiautomorphisms and $a \Cup b$ is the supremum of $\{a, b\}$. Indeed, (11) and (12) capture the following strengthening of (13):

$$y \leq z \quad \Rightarrow \quad z \to x \leq y \to x \quad \text{and} \quad z \rightsquigarrow x \leq y \rightsquigarrow x,$$

a double application of which entails that whenever u is an upper bound of $\{a, b\}$, then $a \Cup b = (a \to b) \rightsquigarrow b \leq (u \to b) \rightsquigarrow b = u \Cup b = u$, proving $a \Cup b = \sup\{a, b\}$.

Remark 3 In the light of Proposition 2 we may think of algebras of type $(2, 2, 0, 0)$ satisfying the equations (4)–(9) as posets with sectional antiautomorphisms. More precisely, if we are given an algebra $\mathcal{A}$, then $\mathcal{A}^P$ is a poset with sectional antiautomorphisms from which $\mathcal{A}$ could be retrieved by means of the operation $\Cup$ defined by (10). Thus *if we use* $\Cup$ *in constructing* $\mathcal{A}^{PA} := (\mathcal{A}^P)^A$ *from* $\mathcal{A}^P$ by Proposition 1, then $\mathcal{A}^{PA} = \mathcal{A}$. However, the correspondence between the class of algebras satisfying (4)–(9) and the class of posets with sectional antiautomorphisms given by the assignments $\mathcal{A} \mapsto \mathcal{A}^P$ and $\mathcal{P} \mapsto \mathcal{P}^A$ is *not* one-one. The reason is that in defining the arrows $\to, \rightsquigarrow$ we first have to fix the join-like operation $\sqcup$, which is not uniquely determined by the underlying order of $\mathcal{A}^P$ unless it is a lattice, and hence $\sqcup$ can differ from $\Cup$, in which case $\mathcal{A}^{PA} \neq \mathcal{A}$. Therefore, one poset with sectional antiautomorphisms in general gives arise to several distinct algebras satisfying (4)–(9) (see Example 1).

On the other hand, an easy inspection shows that for every poset with sectional antiautomorphisms $\mathcal{P}$ we have $\mathcal{P}^{AP} := (\mathcal{P}^A)^P = \mathcal{P}$.

In what follows, in addition to the 'arrows' $\to, \rightsquigarrow$ we will alternatively work with the operations $\oplus, \boxplus, ^-, ^\sim$ defined by

$$\begin{aligned} x^- &:= x \to 0 = \gamma_0(x), & x^\sim &:= x \rightsquigarrow 0 = \delta_0(x), \\ x \oplus y &:= x^\sim \to y = \gamma_y(x^\sim \sqcup y), & x \boxplus y &:= y^- \rightsquigarrow x = \delta_x(x \sqcup y^-). \end{aligned} \tag{14}$$

We should notice that $x \to y = x^- \oplus y$ and $x \rightsquigarrow y = y \boxplus x^\sim$.

2 Pseudo-Effect Algebras

Definition 2 [4] A *pseudo-effect algebra* is a structure $(E, +, 0, 1)$, where $+$ is a partial binary operation on E, satisfying the following conditions:

(PE1) $+$ is associative, in the sense that $(a+b)+c$ is defined if and only if $a+(b+c)$ is defined, and in this case $(a+b)+c=a+(b+c)$;
(PE2) for every $a\in E$ there exist unique $a^-, a^\sim \in E$ such that $a^-+a=1=a+a^\sim$;
(PE3) if $a+b$ is defined, then $a+b=x+a=b+y$ for some $x,y\in E$;
(PE4) if $a+1$ or $1+a$ is defined, then $a=0$.

Every pseudo-effect algebra $(E,+,0,1)$ becomes a bounded poset when equipped with $\leq$ which is given by

$$a\leq b \quad \text{iff} \quad b=x+a \text{ for some } x\in E \quad \text{iff} \quad b=a+y \text{ for some } y\in E.$$

These equivalences determine two partial subtractions $\backslash$ and $/$ on E: $b\backslash a$ is defined and equals x iff $b=x+a$, and a/b is defined and equals y iff $b=a+y$. Thus both $b\backslash a$ and a/b are defined iff $a\leq b$, and then $(b\backslash a)+a=b=a+(a/b)$. It is also clear that for each $a\in E$ we have $1\backslash a=a^-$ and $a/1=a^\sim$.

If the underlying poset $(E,\leq)$ happens to be is a lattice, then $(E,+,0,1)$ is called a *lattice pseudo-effect algebra*.

Let us recall some basic properties of pseudo-effect algebras (see [4], Lemmata 1.4 and 1.6) that we will use in calculations:

$$a^{-\sim}=a=a^{\sim-}; \qquad 1^-=0=1^\sim; \qquad a+0=a=0+a;$$

$$a+b=c \text{ iff } a^\sim=b+c^\sim \text{ iff } b^-=c^-+a; \tag{15}$$

$$a+b \text{ is defined iff } a\leq b^- \text{ iff } a^\sim\geq b;$$

$$a\leq b \text{ iff } a^-\geq b^- \text{ iff } a^\sim\geq b^\sim; \tag{16}$$

$$\text{If } a\leq b, \text{ then } b\backslash a=(a+b^\sim)^- \text{ and } a/b=(b^-+a)^\sim;$$

$$\text{If } b+c \text{ is defined, then } a\leq b \text{ iff } a+c \text{ is defined and } a+c\leq b+c; \tag{17}$$

$$\text{If } c+b \text{ is defined, then } a\leq b \text{ iff } c+a \text{ is defined and } c+a\leq c+b. \tag{18}$$

We first characterize pseudo-effect algebras as certain posets with sectional antiautomorphisms.

Lemma 1 *The following conditions are equivalent in posets with sectional antiautomorphisms*:

$$x\leq y\leq z \quad \Rightarrow \quad \delta_{\gamma_x(z)}(\gamma_x(y))=\gamma_y(z), \tag{19}$$

$$x\leq y\leq z \quad \Rightarrow \quad \gamma_{\delta_x(z)}(\delta_x(y))=\delta_y(z), \tag{20}$$

$$x\leq y \quad \& \quad \gamma_x(y)\leq z \quad \Rightarrow \quad \delta_{\gamma_x(y)}(z)=\gamma_{\delta_x(z)}(y), \tag{21}$$

$$x\leq z \quad \& \quad \delta_x(z)\leq y \quad \Rightarrow \quad \delta_{\gamma_x(y)}(z)=\gamma_{\delta_x(z)}(y). \tag{21'}$$

Proof (19) $\Rightarrow$ (20) If $x\leq y\leq z$, then $x\leq\delta_x(z)\leq\delta_x(y)$, which yields $\gamma_{\delta_x(z)}(\delta_x(y))=\delta_{\gamma_x(\delta_x(y))}(\gamma_x(\delta_x(z)))=\delta_y(z)$.

(20) $\Rightarrow$ (21) Let $x\leq y$ and $\gamma_x(y)\leq z$. Then $x\leq\gamma_x(y)\leq z$ implies $\delta_{\gamma_x(y)}(z)=\gamma_{\delta_x(z)}(\delta_x(\gamma_x(y)))=\gamma_{\delta_x(z)}(y)$.

(21) $\Leftrightarrow$ (21′) Actually, the condition (21′) is merely a reformulation of (21) because the assumption that $x\leq y$ and $\gamma_x(y)\leq z$ is the same as $x\leq z$ and $\delta_x(z)\leq y$. Indeed, if $x\leq y$

and $\gamma_x(y) \le z$, then $x \le z$ and $\delta_x(z) \le \delta_x(\gamma_x(y)) = y$, and conversely, if $x \le z$ and $\delta_x(z) \le y$, then $x \le y$ and $\gamma_x(y) \le \gamma_x(\delta_x(z)) = z$.

(21′) ⇒ (19) Let $x \le y \le z$. Then $x \le \gamma_x(y)$ and $\delta_x(\gamma_x(y)) = y \le z$ together entail $\delta_{\gamma_x(z)}(\gamma_x(y)) = \gamma_{\delta_x(\gamma_x(y))}(z) = \gamma_y(z)$. □

Proposition 3 *Let $\mathcal{E} = (E, +, 0, 1)$ be a pseudo-effect algebra. For every $e \in E$ the maps $\gamma_e, \delta_e : [e, 1] \to [e, 1]$ defined by $\gamma_e(x) = x^- + e$ and $\delta_e(x) = e + x^\sim$ are antiautomorphisms which are inverses of one another. Thus $\mathcal{E}^P = (E, \le, (\gamma_e, \delta_e)_{e\in E}, 0, 1)$, where $\le$ is the underlying order of $\mathcal{E}$, is a poset with sectional antiautomorphisms which in addition satisfies the conditions* (19)–(21).

Proof It is obvious that $\gamma_e, \delta_e : [e, 1] \to [e, 1]$ are well-defined because for any $x \in [e, 1]$, $x^- + e$ and $e + x^\sim$ exist and are greater or equal to e. Both γ_e and δ_e are antitone: If $e \le x \le y$, then $\gamma_e(x) = x^- + e \ge y^- + e = \gamma_e(y)$ and $\delta_e(x) = e + x^\sim \ge e + y^\sim = \delta_e(y)$ by (16), (17) and (18). Moreover, by (15) we have $\delta_e(\gamma_e(x)) = e + (x^- + e)^\sim = x$ and $\gamma_e(\delta_e(x)) = (e + x^\sim)^- + e = x$. Hence γ_e, δ_e are mutually inverse antiautomorphisms on $[e, 1]$. Finally, if $x \le y$ and $\gamma_x(y) = y^- + x \le z$, then $\delta_{\gamma_x(y)}(z) = (y^- + x) + z^\sim = y^- + (x + z^\sim) = \gamma_{\delta_x(z)}(y)$, which is just (21). □

Proposition 4 *Let $\mathcal{P} = (P, \le, (\gamma_p, \delta_p)_{p\in P}, 0, 1)$ be a poset with sectional antiautomorphisms satisfying the equivalent conditions* (19)–(21), *and let us define the partial addition $+$ on P as follows*:

$$x + y \text{ is defined iff } \delta_0(x) \ge y, \text{ and in this case } x + y := \gamma_y(\delta_0(x)).$$

Then $\mathcal{P}^E = (P, +, 0, 1)$ is a pseudo-effect algebra whose underlying order is just the initial order $\le$, and where $x^- = \gamma_0(x)$ and $x^\sim = \delta_0(x)$.

Proof First of all we observe that $x + y$ is defined iff $\delta_0(x) \ge y$ iff $x \le \gamma_0(y)$, and

$$x + y := \gamma_y(\delta_0(x)) = \delta_x(\gamma_0(y)).$$

Indeed, $\delta_0(x) \ge y$ is clearly equivalent to $x \le \gamma_0(y)$, and $0 \le y \le \delta_0(x)$ implies $\gamma_y(\delta_0(x)) = \delta_{\gamma_0(\delta_0(x))}(\gamma_0(y)) = \delta_x(\gamma_0(y))$ by (19).

(PE1) Let $(x + y) + z$ be defined, i.e., $\delta_0(x) \ge y$ and $x + y \le \gamma_0(z)$. Since $y \le \gamma_y(\delta_0(x)) = x + y \le \gamma_0(z)$, also $y + z$ exists and we have $y + z = \delta_y(\gamma_0(z)) \le \delta_y(\gamma_y(\delta_0(x))) = \delta_0(x)$, so $x + (y + z)$ is defined, too. Conversely, assume $x + (y + z)$ exists, i.e., $y \le \gamma_0(z)$ and $\delta_0(x) \ge y + z$. Then $y \le \delta_y(\gamma_0(z)) = y + z \le \delta_0(x)$ entails the existence of $x + y$ and we have $x + y = \gamma_y(\delta_0(x)) \le \gamma_y(\delta_y(\gamma_0(z))) = \gamma_0(z)$, hence $(x + y) + z$ is defined.

Thus $(x + y) + z$ exists iff so does $x + (y + z)$, in which case, using

$$\delta_y(x + y) = \delta_y(\gamma_y(\delta_0(x))) = \delta_0(x),$$

from $y \le \gamma_y(\delta_0(x)) = x + y \le \gamma_0(z)$ we obtain

$$(x + y) + z = \delta_{x+y}(\gamma_0(z)) = \gamma_{\delta_y(\gamma_0(z))}(\delta_y(x + y)) = \gamma_{y+z}(\delta_0(x)) = x + (y + z)$$

by (20). This settles (PE1).

(PE2) It is plain that $\gamma_0(x) + x$ is defined and $\gamma_0(x) + x = \delta_{\gamma_0(x)}(\gamma_0(x)) = 1$. On the other hand, if $y + x$ exists and equals 1, then $y \le \gamma_0(x)$ and $1 = y + x = \delta_y(\gamma_0(x))$, which is possible only if $\gamma_0(x) = y$. Hence $x^- = \gamma_0(x)$. Analogously, $x^\sim = \delta_0(x)$.

(PE3) Let $x+y$ be defined. With $u:=\gamma_0(\delta_x(x+y))$ and $v:=\delta_0(\gamma_y(x+y))$ we have $u+x=x+y=y+v$. Indeed, $u+x=\gamma_x(\delta_0(u))=\gamma_x(\delta_x(x+y))=x+y$ and $y+v=\delta_y(\gamma_0(v))=\delta_y(\gamma_y(x+y))=x+y$.

(PE4) If $x+1$ or $1+x$ is defined, then $x\le\gamma_0(1)=0$ or $0=\delta_0(1)\ge x$, so that $x=0$.

There remains to show that the initial partial order $\le$ on P is the underlying order of the pseudo-effect algebra $\mathcal{P}^E$, i.e., $a\le b$ iff $b=a+x$ for some $x\in P$. But this is almost evident because if $a\le b$, then $a+\delta_0(\gamma_a(b))$ exists for $a\le\gamma_a(b)=\gamma_0(\delta_0(\gamma_a(b)))$, and $a+\delta_0(\gamma_a(b))=\delta_a(\gamma_0(\delta_0(\gamma_a(b))))=\delta_a(\gamma_a(b))=b$, and conversely, if $b=a+x=\delta_a(\gamma_0(x))$, then trivially $b\ge a$. □

Theorem 1 *The correspondence between pseudo-effect algebras and posets with sectional antiautomorphisms satisfying* (19)–(21) *as established in Propositions* 3 *and* 4 *is one-one.*

Proof (1) Let $\mathcal{E}=(E,+,0,1)$ be a pseudo-effect algebra, $\mathcal{E}^P=(E,\le,(\gamma_e,\delta_e)_{e\in E},0,1)$ the poset with sectional antiautomorphisms from Proposition 3 and $\mathcal{E}^{PE}=(E,\star,0,1)$ the effect algebra associated to $\mathcal{E}^P$ by Proposition 4. We know that $\le$ is the underlying order of $\mathcal{E}$ as well as of $\mathcal{E}^{PE}$. Moreover, $a\star b$ is defined iff $a\le\gamma_0(b)=b^-$ iff $a+b$ is defined, and in this case $a\star b=\gamma_b(\delta_0(a))=a^{\sim -}+b=a+b$. Hence $\mathcal{E}^{PE}=\mathcal{E}$.

(2) Given $\mathcal{P}=(P,\le,(\gamma_p,\delta_p)_{p\in P},0,1)$ a poset with sectional antiautomorphisms satisfying (19)–(21), let $\mathcal{P}^E=(P,+,0,1)$ be the pseudo-effect algebra constructed by Proposition 4, and $\mathcal{P}^{EP}=(P,\preceq,(\epsilon_p,\eta_p)_{p\in P},0,1)$ be the poset with sectional antiautomorphisms associated to $\mathcal{P}^E$ according to Proposition 3. It follows from Proposition 4 that $\preceq$ and $\le$ coincide. Also, for every $p\in P$ and $x\in[p,1]$ we have $\epsilon_p(x)=x^-+p=\gamma_p(\delta_0(\gamma_0(x)))=\gamma_p(x)$ and $\eta_p(x)=p+x^\sim=\delta_p(\gamma_0(\delta_0(x)))=\delta_p(x)$. Thus $\mathcal{P}^{EP}=\mathcal{P}$. □

Now let $\mathcal{E}=(E,+,0,1)$ be a pseudo-effect algebra. We may associate a total algebra to $\mathcal{E}$ by making the corresponding poset with sectional antiautomorphisms $\mathcal{E}^P=(E,\le,(\gamma_e,\delta_e)_{e\in E},0,1)$ into the algebra $\mathcal{E}^{PA}=(E,\to,\rightsquigarrow,0,1)$ in accordance with Proposition 1. Thus we first equip E with the operation $\sqcup$ which obeys the rule (3), and then define the total operations $\to$ and $\rightsquigarrow$ as follows:

$$x\to y:=\gamma_y(x\sqcup y)=(x\sqcup y)^-+y,\qquad x\rightsquigarrow y:=\delta_y(x\sqcup y)=y+(x\sqcup y)^\sim.\tag{22}$$

We shall briefly write $\mathcal{E}^A$ in place of $\mathcal{E}^{PA}$.

Theorem 2 *Let* $\mathcal{E}=(E,+,0,1)$ *be a pseudo-effect algebra and let the operations* $\to,\rightsquigarrow$ *be defined by* (22). *Then the algebra* $\mathcal{E}^A=(E,\to,\rightsquigarrow,0,1)$ *satisfies the identities* (4)–(9) *and the quasi-identities*

$$x\le y\le z\quad\Rightarrow\quad(y\to x)\rightsquigarrow(z\to x)=z\to y,\tag{23}$$

$$x\le y\le z\quad\Rightarrow\quad(y\rightsquigarrow x)\to(z\rightsquigarrow x)=z\rightsquigarrow y,\tag{24}$$

$$x\le y\quad\&\quad y\to x\le z\quad\Rightarrow\quad z\rightsquigarrow(y\to x)=y\to(z\rightsquigarrow x).\tag{25}$$

Proof It is clear that $\mathcal{E}^A$ satisfies (4)–(9) by Proposition 1, and the quasi-identities (23)–(25) are precisely the conditions (19)–(21) expressed by means of $\to,\rightsquigarrow$. □

The total operations $\boxplus$ and $\oplus$ from (14) are given by

$$x\boxplus y:=y^-\rightsquigarrow x=x+(x\sqcup y^-)^\sim,\qquad x\oplus y:=x^\sim\to y=(x^\sim\sqcup y)^-+y.$$

Obviously, we have $x \boxplus y = x \oplus (x \sqcup y^-)^\sim$ and $x \oplus y = (x^\sim \sqcup y)^- \boxplus y$, and if $x + y$ exists in $\mathcal{E}$, then both $x \boxplus y$ and $x \oplus y$ are equal to $x + y$. It is also easily seen that (25) [resp. the condition (21)] is equivalent to the quasi-identity

$$x \leq y^- \quad \& \quad x \oplus y \leq z^- \quad \Rightarrow \quad (x \oplus y) \boxplus z = x \oplus (y \boxplus z),$$

which is further equivalent to the quasi-identity

$$x \leq y^- \quad \& \quad x \oplus y \leq z^- \quad \Rightarrow \quad (x \oplus y) \oplus z = x \oplus (y \oplus z).$$

By combining Propositions 2, 3 and 4 we obtain:

Theorem 3 *Given* $\mathcal{A} = (A, \to, \rightsquigarrow, 0, 1)$ *an algebra satisfying* (4)–(9), *let us define the partial addition* $+$ *as follows*:
$x + y$ *is defined and equal to* $x \oplus y = x^\sim \to y$ $(= y^- \rightsquigarrow x = x \boxplus y)$ *iff* $x^\sim \geq y$ (*iff* $x \leq y^-$). *Then* $\mathcal{A}^E = (A, +, 0, 1)$ *is a pseudo-effect algebra if and only if* $\mathcal{A}$ *satisfies the equivalent conditions* (23)–(25).

We have $\mathcal{E}^{AE} = \mathcal{E}$, but the assignments $\mathcal{A} \mapsto \mathcal{A}^E$ and $\mathcal{E} \mapsto \mathcal{E}^A$ do not establish a one-one correspondence because the passage from $\mathcal{E}$ to $\mathcal{E}^A$ strongly depends on how the 'join-like' operation $\sqcup$ has been defined, so that it can well happen that $\mathcal{A}^{EA}$ differs from $\mathcal{A}$ (see Remark 3).

3 Compatibility in Lattice Pseudo-Effect Algebras

Compatibility of elements plays an important role in (lattice) effect algebras. Among others, Riečanová [8] proved that every lattice effect algebra is a union of its blocks (= maximal subsets of mutually compatible elements) and that these blocks are MV-algebras.

Dvurečenskij and Vetterlein [3, 6] defined several kinds of compatibilities in pseudo-effect algebras:

Definition 3 [3, 6] Let $(E, +, 0, 1)$ be a pseudo-effect algebra. Then $a, b \in E$ are

(i) *strongly compatible* (in symbols $a \stackrel{c}{\longleftrightarrow} b$) if there exist $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$, $a_1 + b_1 + c = b_1 + a_1 + c$ and $a_1 \wedge b_1 = 0$;
(ii) *compatible* (in symbols $a \leftrightarrow b$) if there exist $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$ and $a_1 + b_1 + c = b_1 + a_1 + c$;
(iii) *weakly compatible* (in symbols $a \stackrel{w}{\longleftrightarrow} b$) if there exist $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$ and both $a_1 + b_1 + c$ and $b_1 + a_1 + c$ are defined;
(iv) *ultra weakly compatible* (in symbols $a \stackrel{uw}{\longleftrightarrow} b$) if there exist $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$, and $a_1 + b_1 + c$ or $b_1 + a_1 + c$ is defined.

By [6], Theorem 3.8, if $(E, +, 0, 1)$ is a *lattice* pseudo-effect algebra, then $a \stackrel{c}{\longleftrightarrow} b$ iff $a \leftrightarrow b$ iff $a \stackrel{w}{\longleftrightarrow} b$ for all $a, b \in E$. Obviously, $a \leftrightarrow b$ yields $a \stackrel{uw}{\longleftrightarrow} b$, but the reverse implication fails to be true.

Let us recall that two elements a, b in an effect algebra $(E, +, 0, 1)$ are said to be *compatible* provided that there exist $a_1, b_1, c \in E$ so that $a = a_1 + c$, $b = b_1 + c$ and $a_1 + b_1 + c$ is defined in E. It is therefore apparent that in case that $(E, +, 0, 1)$ is an effect algebra the above concepts of compatibility, weak compatibility and ultra weak compatibility coincide, i.e., $a \leftrightarrow b$ iff $a \overset{w}{\longleftrightarrow} b$ iff $a \overset{uw}{\longleftrightarrow} b$.

In [1] we have proved that if $(E, +, 0, 1)$ is a lattice effect algebra, then

$$a \leftrightarrow b \quad \text{if and only if} \quad a \oplus b = b \oplus a. \tag{26}$$

Our final objective is to characterize pairs of compatible elements in pseudo-effect algebras in terms of the total operations we have defined in Sect. 2. We start with ultra weak compatibility:

Theorem 4 *Let $(E, +, 0, 1)$ be a lattice-ordered pseudo-effect algebra. For all $a, b \in E$ the following are equivalent*:

(i) $a \overset{uw}{\longleftrightarrow} b$;
(ii) $a \rightsquigarrow b = b^\sim \to a^\sim$ *or* $b \rightsquigarrow a = a^\sim \to b^\sim$;
(iii) $a \to b = b^- \rightsquigarrow a^-$ *or* $b \to a = a^- \rightsquigarrow b^-$;
(iv) $b \oplus a^\sim = b \boxplus a^\sim$ *or* $a \oplus b^\sim = a \boxplus b^\sim$;
(v) $a^- \oplus b = a^- \boxplus b$ *or* $b^- \oplus a = b^- \boxplus a$.

Proof By [6], Proposition 3.10,

$$\begin{aligned} a \overset{uw}{\longleftrightarrow} b \quad &\text{iff} \quad a\backslash(a \wedge b) = (a \vee b)\backslash b \text{ or } b\backslash(a \wedge b) = (a \vee b)\backslash a, \\ &\text{iff} \quad (a \wedge b)/a = b/(a \vee b) \text{ or } (a \wedge b)/b = a/(a \vee b). \end{aligned} \tag{27}$$

Further, for every $a, b \in E$ we have $(a \vee b)\backslash b = (b + (a \vee b)^\sim)^- = (a \rightsquigarrow b)^- = (b \boxplus a^\sim)^-$ and $a\backslash(a \wedge b) = ((a \wedge b) + a^\sim)^- = ((a^\sim \vee b^\sim)^- + a^\sim)^- = (b^\sim \to a^\sim)^- = (b \oplus a^\sim)^-$, hence

$$a\backslash(a \wedge b) = (a \vee b)\backslash b \quad \text{iff} \quad a \rightsquigarrow b = b^\sim \to a^\sim \quad \text{iff} \quad b \boxplus a^\sim = b \oplus a^\sim, \tag{28}$$

and similarly, $b/(a \vee b) = ((a \vee b)^- + b)^\sim = (a \to b)^\sim = (a^- \oplus b)^\sim$ and $(a \wedge b)/a = (a^- + (a \wedge b))^\sim = (a^- + (a^- \vee b^-)^\sim)^\sim = (b^- \rightsquigarrow a^-)^\sim = (a^- \boxplus b)^\sim$, so that

$$(a \wedge b)/a = b/(a \vee b) \quad \text{iff} \quad a \to b = b^- \rightsquigarrow a^- \quad \text{iff} \quad a^- \oplus b = a^- \boxplus b. \tag{29}$$

The equivalence of (i)–(v) now easily follows by combining (27), (28) and (29). □

If we want to describe compatibility, it suffices to replace 'or' by 'and':

Corollary 1 *Let $(E, +, 0, 1)$ be a lattice-ordered pseudo-effect algebra. For all $a, b \in E$ the following are equivalent*:

(i) $a \leftrightarrow b$;
(ii) $a \rightsquigarrow b = b^\sim \to a^\sim$ *and* $b \rightsquigarrow a = a^\sim \to b^\sim$;
(iii) $a \to b = b^- \rightsquigarrow a^-$ *and* $b \to a = a^- \rightsquigarrow b^-$;
(iv) $b \oplus a^\sim = b \boxplus a^\sim$ *and* $a \oplus b^\sim = a \boxplus b^\sim$;
(v) $a^- \oplus b = a^- \boxplus b$ *and* $b^- \oplus a = b^- \boxplus a$.

Proof This is a direct consequence of Theorem 4 because

$$a \leftrightarrow b \quad \text{iff} \quad a\backslash(a \wedge b) = (a \vee b)\backslash b \text{ and } b\backslash(a \wedge b) = (a \vee b)\backslash a,$$

$$\text{iff} \quad (a \wedge b)/a = b/(a \vee b) \text{ and } (a \wedge b)/b = a/(a \vee b)$$

by [6], Proposition 3.6. □

Theorem 5 *Let* $(E, +, 0, 1)$ *be a lattice pseudo-effect algebra that satisfies the following additional condition*:

$$x \leftrightarrow y \quad \Rightarrow \quad x \leftrightarrow y^- \text{ and } x \leftrightarrow y^\sim. \tag{30}$$

Then, for all $a, b \in E$, *we have* $a \leftrightarrow b$ *iff* $a \overset{uw}{\longleftrightarrow} b$ *iff* $a \oplus b = a \boxplus b$ *iff* $a^\sim \to b = b^- \rightsquigarrow a$.

Proof By [6], Proposition 4.8, the condition (30) entails that all the aforementioned types of compatibility coincide, thus $a \overset{uw}{\longleftrightarrow} b$ is the same as $a \leftrightarrow b$. Consequently, if $a \leftrightarrow b$, then also $a \leftrightarrow b^-$ and $a^- \leftrightarrow b$, which yields $a \oplus b = a \boxplus b$ and $b \oplus a = b \boxplus a$ by Corollary 1 (iv). On the other hand, if $a \oplus b = a \boxplus b$ (or $b \oplus a = b \boxplus a$), then (iv) of Theorem 4 implies $a \overset{uw}{\longleftrightarrow} b^-$, which is equivalent to $a \leftrightarrow b^-$, and hence $a \leftrightarrow b$ by (30). □

The last theorem generalizes (26) because the condition (30) automatically holds if $(E, +, 0, 1)$ is a lattice effect algebra, and in this case we have $a \oplus b = (a^- \vee b)^- + b$ and $a \boxplus b = a + (a \vee b^-)^- = (a \vee b^-)^- + a = b \oplus a$.

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